

## Nonlinear Duality and Best Approximations in Metric Linear Spaces

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### I. INTRODUCTION AND NOTATION

The dual space is a very powerful tool in the theory of best approximations in normed linear spaces and—a well known fact—it can be used for characterization as well as computation of best approximations. In metric spaces, however, this dual often consists of the zero functional only so that, in general, the rich duality theory is not applicable. Many authors have given characterizations of best approximations in special metric linear spaces by linear or nonlinear functionals (e.g., [1, 2, 7, 9–11, 13]), but there does not seem to be an investigation on the functional analytic background of the “dual spaces” used.

In this paper we shall construct a “nonlinear dual space” and relate some of its functional analytic properties to the corresponding ones of linear dual spaces of normed linear spaces. The nonlinear dual can be used to characterize best approximations (from linear subspaces) and in special cases it can be used to compute best approximations with a Remez-type algorithm (for the latter see [10]).

Let  $X$  be a real metric linear space with translation invariant metric  $d$ , and  $q: X \rightarrow \mathbb{R}_+$  the canonical quasinorm which is defined by

$$q(x) := d(x, 0)$$

for every  $x \in X$ . The mapping  $q$  has all the properties of a norm but positive homogeneity and this is the reason why approximation theory in metric linear spaces is so much different from the theory in normed spaces. Unit balls are neither convex nor even connected in general and there are examples of spaces with one-dimensional nonproximal subspaces (see [2]).

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If  $l$  is a real valued function defined on  $X$ , we denote by

$$\|l\| := \sup \left\{ \frac{|l(x)|}{q(x)}; x \neq 0 \right\}$$

the norm of that function whenever this supremum is finite. We say that  $l$  is Lipschitz continuous (with Lipschitz constant  $k$ ) if for all  $x, y \in X$ ,

$$|l(x) - l(y)| \leq k \cdot q(x - y)$$

holds,  $l$  is called odd (even) if

$$-l(x) = l(-x) \quad (l(x) = l(-x))$$

holds for all  $x \in X$ .

Note that for Lipschitz continuous odd functions with Lipschitz constant  $k > 0$  the norm is always well defined and satisfies

$$\|l\| \leq k.$$

Next we denote by

$$\text{Lip}_k(X)$$

the set of all Lipschitz continuous odd functions with Lipschitz constant  $k$ . This set is easily seen to be convex but it is not closed under the linear operations of addition and scalar multiplication and hence not a linear space. But if we take

$$\text{Lip}(X) := \bigcup_{k > 0} \text{Lip}_k(X),$$

then the pair  $(\text{Lip}(X), \|\cdot\|)$  becomes a normed linear space. But as the set of functions we are constructing is supposed to be an equivalent to the dual space of a normed linear space and since  $\text{Lip}(X)$  is not a Banach space in general (an outline of the proof of this fact will be briefly sketched after Theorem 7), we fix  $k > 0$  and call  $\text{Lip}_k(X)$  the “nonlinear dual space with index  $k$ .” This set is always complete in the topology induced by  $\|\cdot\|$  and, as will be seen in the next chapter, is closely connected with the dual spaces of normed linear spaces.

Finally if  $V \subset X$  is a subspace,  $\text{Lip}_k(V)$  and  $\text{Lip}(V)$  are defined analogously with  $V$  in place of  $X$ . When it seems necessary, we denote by  $\|\cdot\|_V$  the norm in  $\text{Lip}(V)$  and by  $\|\cdot\|_X$  the norm in  $\text{Lip}(X)$ , respectively.

II. SOME PROPERTIES OF THE NONLINEAR DUAL SPACE

In this chapter we collect some properties of the nonlinear dual spaces which show that they are in fact very similar to duals of normed linear spaces. In particular analogous results to the theorems of Alaoglu–Bourbaki, Krein–Milman, and Hahn–Banach will be provided.

To do so, we introduce a weak topology  $T_k$  on the nonlinear dual space. Let  $k > 0$  be fixed. For every finite set  $a \subset X$ , every  $\varepsilon > 0$ , and every  $l_0 \in \text{Lip}_k(X)$  we define

$$U_{a,\varepsilon}^k(l_0) := \bigcap_{x \in a} \{l \in \text{Lip}_k(X); l(x) - l_0(x) < \varepsilon\}.$$

The collection of all these sets will form a subbasis for the topology  $T_k$ . Likewise we define

$$U_{a,\varepsilon}(l_0) := \bigcap_{x \in a} \{l \in \text{Lip}(X); l(x) - l_0(x) < \varepsilon\}$$

for  $l_0 \in \text{Lip}(X)$  as a subbasis for a topology  $T$  on  $\text{Lip}(X)$ . We then have for  $k > 0$ ,

LEMMA 1.  $T_k$  and  $T$  are locally convex Hausdorff topologies. A net of functionals  $(l_\delta) \subset \text{Lip}_k(X)$  (resp.  $(l_\delta) \subset \text{Lip}(X)$ ) converges to  $l_0 \in \text{Lip}_k(X)$  (resp.  $l_0 \in \text{Lip}(X)$ ) in the topology  $T_k$  (resp.  $T$ ) if and only if it converges pointwise.

The proof is not difficult and omitted here, it can be performed like the proof of Satz (I.2.1) in [10]. We shall now provide a simple fact about Lipschitz continuous functions with Lipschitz constant  $k$  which will be useful in the sequel.

LEMMA 2. The pointwise limit of a net  $(l_\delta)$  of Lipschitz continuous functions with Lipschitz index  $k$  is again Lipschitz continuous with the same index  $k$ .

*Proof.* Let  $(l_\delta, \delta \in D)$  be a net of Lipschitz continuous functions (all with Lipschitz index  $k$ ) converging pointwise to  $l_0$ ; here  $D$  is an ordered set and “ $>$ ” is the order relation in  $D$ . Note that  $l_0(x)$  exists for all  $x \in X$  since

$$|l_0(x)| \leq |l_\delta(x)| + |l_\delta(x) - l_0(x)|$$

for all  $\delta \in D$ . If now  $\varepsilon > 0$  is given and  $x, y \in X$  such that  $q(x - y) > 0$  (if  $x = y$  there is nothing to be shown), we may choose  $\delta_0 = \delta_0(\varepsilon) \in D$  such that for all  $\delta > \delta_0$ , we have

$$|l_0(x) - l_\delta(x)| \leq \frac{k \cdot \varepsilon}{2} \cdot q(x - y) \quad \text{and} \quad |l_0(y) - l_\delta(y)| \leq \frac{k \cdot \varepsilon}{2} \cdot q(x - y)$$

and  $\delta > \delta_0$  yields

$$\begin{aligned} & |l_0(x) - l_0(y)| \\ & \leq |l_0(x) - l_\delta(x)| + |l_\delta(x) - l_\delta(y)| + |l_\delta(y) - l_0(y)| \\ & \leq k \cdot (1 + \varepsilon) \cdot q(x - y). \end{aligned}$$

This completes the proof since  $\varepsilon > 0$  was arbitrary.  $\blacksquare$

We denote by  $\text{ext}(A)$  the set of all extreme points of a set  $A$  and by  $\text{co}(B)$  the convex hull of a set  $B$  and prove a Krein–Milman-type theorem for  $\text{Lip}_k(X)$ . Part (a) of the proof is a slight modification of the one found in [5, p. 439].

**THEOREM 3.** *If  $k > 0$  is fixed and  $K \subset \text{Lip}_k(X)$  is convex and compact relative to the topology  $T_k$ , then*

- (a)  $\text{ext}(K) \neq \emptyset$ ,
- (b)  $K = \overline{\text{co}}(\text{ext}(K))$ .

*Proof.* To prove (a) let  $E$  be the nonvoid family of all closed extremal subsets of  $K$  ordered by inclusion. If  $E_1$  is a totally ordered subfamily of  $E$ , then by compactness  $\bigcap E_1$  is nonempty, extremal since it is an intersection of extremal sets, and is a lower bound for  $E_1$ . Hence by Zorn's lemma, there is a minimal element  $A_0$  in  $E$ . We show that  $A_0$  contains one element only and hence is an extreme point.

To this aim assume that  $l_1$  and  $l_2$  are two distinct elements of  $A_0$ ; i.e., there is a point  $z \in X$  such that

$$l_1(z) \neq l_2(z).$$

But then

$$A_1 := \{l \in A_0; l(z) = \inf\{a(z), a \in A_0\}\}$$

is a proper subset of  $A_0$ . On the other hand  $L_1, L_2 \in K$ ,  $\lambda \in (0, 1)$  and  $\lambda L_1 + (1 - \lambda) L_2 \in A_1$  imply

$$L_1, L_2 \in A_0,$$

since  $A_0$  is extremal and from the definition of  $A_1$ , we conclude

$$L_1, L_2 \in A_1.$$

This shows that  $A_1$  is a proper extremal subset of  $A_0$  which is a contradiction to  $A_0$  being minimal.

To prove (b), we first show that  $K$  is also a compact subset of  $\text{Lip}(X)$  relative to the topology  $T$ . For, if

$$\bigcup_{\beta \in B} A_\beta$$

is an open cover of  $K$  in  $\text{Lip}(X)$ , then by definition

$$\bigcup_{\beta \in B} (A_\beta \cap \text{Lip}_k(X))$$

is an open cover of  $K$  in  $\text{Lip}_k(X)$  which has a finite subcover, say

$$\bigcup_{i=1}^N (A_{\beta(i)} \cap \text{Lip}_k(X)).$$

But then of course

$$\bigcup_{i=1}^N A_{\beta(i)}$$

is a finite subcover of  $K$  in  $\text{Lip}_k(X)$ .

Since  $\text{Lip}(X)$  is a locally convex topological linear space with topology  $T$ , we know that

$$K = \overline{\text{co}}^{(T)}(\text{ext}(K)),$$

where the closure is taken in the topology  $T$ . But by Lemma 1, convergence of a net  $(I_\delta)$  in the topology  $T$  is equivalent to pointwise convergence in  $\mathbb{R}$ , so that Lemma 2 and

$$\text{co}(\text{ext}(K)) \subset K \subset \text{Lip}_k(X)$$

imply

$$\overline{\text{co}}^{(T)}(\text{ext}(K)) \subset K \subset \text{Lip}_k(X).$$

Since  $T_k$  is the restriction of  $T$  to  $\text{Lip}_k(X)$ , we even have

$$\overline{\text{co}}(\text{ext}(K)) \subset K,$$

where the closure is taken relative to  $T_k$ . This proves (b) since by definition

$$K \subset \overline{\text{co}}(\text{ext}(K)). \quad \blacksquare$$

Our next step is to prove an Alaoglu–Bourbaki-type theorem which is a corollary to

**THEOREM 4.** For every  $k > 0$  the nonlinear dual space  $\text{Lip}_k(X)$  is compact relative to the topology  $T_k$ .

*Proof.* For every  $x \in X$ , the interval  $I_x := [-k \cdot q(x), k \cdot q(x)]$  is a compact subset of the real line and by Tychonoff's theorem the product space

$$P := \prod_{x \in X} I_x := \left\{ l: X \rightarrow \bigcup_{x \in X} I_x; l(x) \in I_x \right\}$$

is compact relative to the product topology which is generated by the sets

$$\bigcap_{x \in a} \{ \pi_x^{-1}(U_x); U_x \subset I_x \text{ open, } a \subset X \text{ finite} \};$$

here  $\pi_x: P \rightarrow I_x$  is the projection mapping from  $P$  to the component  $I_x$ . We recall that a net  $(l_\delta, \delta \in D)$   $P$  converges to  $l_0 \in P$  relative to the product topology if and only if it converges pointwise in  $\mathbb{R}$ .

Since we have the inclusion

$$\begin{aligned} P &= \{ l: X \rightarrow \mathbb{R}; |l(x)| \leq k \cdot q(x) \} \\ &\supset \{ l: X \rightarrow \mathbb{R}; \|l\| \leq k \} \supset \text{Lip}_k(X), \end{aligned}$$

it suffices to show that  $\text{Lip}_k(X)$  is closed in  $P$  relative to the product topology.

But since convergence of a net  $(l_\delta, \delta \in D) \subset \text{Lip}_k(X)$  in the product topology to  $l_0 \in P$  is equivalent to pointwise convergence, this immediately yields

$$l_0(-x) = -l_0(x)$$

for all  $x \in X$  as well as

$$\|l_0\| \leq k$$

and together with Lemma 2 we have

$$l_0 \in \text{Lip}_k(X)$$

which completes the proof. ■

We now define for every  $l_0 \in \text{Lip}_k(X)$  and  $\rho > 0$  the norm balls by

$$B_k(l_0, \rho) := \{ l \in \text{Lip}_k(X); \|l_0 - l\| \leq \rho \}.$$

Since  $B_k(l_0, \rho)$  is a closed subset of  $\text{Lip}_k(X)$  relative to the topology  $T_k$ , we have

COROLLARY 5. *The norm unit balls  $B_k(l_0, \rho)$  are  $T_k$ -compact for every  $k > 0$ ,  $l_0 \in \text{Lip}_k(X)$  and  $\rho > 0$ .*

Note that the norm unit balls are convex too, so that Theorem 3 can be applied to  $B_k(l_0, \rho)$  in place of  $K$ .

Another important tool in the theory of normed linear spaces is the Hahn-Banach extension theorem, which has numerous applications in functional analysis as well as optimization and approximation theory. It can be proved (in an analogous version) for the nonlinear dual spaces considered here. Unlike the classical case, the proof here will be constructive; it has to be mentioned, however, that the result is not a generalization of the classical theorem in the case where  $q$  is a norm, since the extension constructed is not linear in general. The extension principle was first used by McShane [8] in 1934. He used the function  $L_s$  below in the case where  $p$  is a norm, and showed that it has the same Lipschitz constant as  $l$ .

THEOREM 6. *Let  $p: X \rightarrow \mathbb{R}$  be even and subadditive with  $p(0) = 0$  and let  $V \subset X$  be a linear subspace. Then every odd function*

$$l: V \rightarrow \mathbb{R}$$

with

$$l(u) - l(v) \leq p(u - v)$$

for all  $u, v \in V$  has an odd extension  $L$  to all of  $X$  satisfying

$$L(x) - L(y) \leq p(x - y)$$

for all  $x, y \in X$ .

*Proof.* For every  $x \in X$ , we define

$$L_s(x) := \sup\{l(v) - p(x - v), v \in V\}$$

and

$$L_i(x) := \inf\{l(u) + p(x - u), u \in V\}.$$

If  $u, v \in V$  and  $x \in X$  are chosen arbitrary, we have

$$l(v) - p(x - v) \leq l(u) + p(u - v) - p(x - v) \leq l(u) + p(x - u),$$

since  $p$  is even and this yields

$$L_s(x) \leq L_i(x)$$

for all  $x \in X$ . Using  $p(0) = 0$  we know that

$$l(v) \leq L_s(v) \leq L_i(v) \leq l(v)$$

holds for all  $v \in V$  and it follows that both  $L_s$  and  $L_i$  are extensions of  $l$ . If  $x, y \in X$  and  $\varepsilon > 0$  are chosen arbitrary, we have

$$\begin{aligned} L_s(x) - L_s(y) &= \sup\{l(v) - p(x - v)\} - \sup\{l(u) - p(y - u)\} \\ &\leq l(v_\varepsilon) - p(x - v_\varepsilon) + \varepsilon - (l(v_\varepsilon) - p(y - v_\varepsilon)) \\ &\leq p(x - y) + \varepsilon \end{aligned}$$

for some  $v_\varepsilon \in V$ ; since  $\varepsilon > 0$  is arbitrary this yields

$$L_s(x) - L_s(y) \leq p(x - y) \quad \text{for all } x, y \in X. \quad (1)$$

The inequality

$$L_i(x) - L_i(y) \leq p(x - y) \quad \text{for all } x, y \in X \quad (2)$$

can be shown analogously.

If we did not want the extension  $L$  to be odd, both  $L_s$  and  $L_i$  would have the desired properties, but since they are not odd in general, we define

$$L(x) := \begin{cases} L_s(x) & \text{if } L_s(x) > 0, \\ 0 & \text{if } L_s(x) \leq 0 \leq L_i(x), \\ L_i(x) & \text{if } L_i(x) < 0. \end{cases}$$

Since  $L_s$  as well as  $L_i$  are extensions of  $l$ , so is  $L$ . It is easy to see that

$$-L_s(x) = L_i(-x) \quad \text{and} \quad -L_i(x) = L_s(-x)$$

for all  $x \in X$  and this implies  $-L(x) = L(-x)$  for all  $x \in X$  and hence  $L$  is odd.

It now remains to show that  $L$  satisfies

$$L(x) - L(y) \leq p(x - y)$$

for all  $x, y \in X$ . For arbitrary  $x, y \in X$ , we consider the four cases:

Case 1.  $L(x) > 0$  and  $L(y) \geq 0$ .

Case 2.  $L(x) > 0$  and  $L(y) < 0$ .

Case 3.  $L(x) \leq 0$  and  $L(y) \geq 0$ .

Case 4.  $L(x) \leq 0$  and  $L(y) < 0$ .



Case 1 follows from (1) and  $L(x) - L(y) \leq L_s(x) - L_s(y)$ . Case 2 follows from (2) and  $L(x) - L(y) \leq L_s(x) - L_i(y) \leq L_i(x) - L_i(y)$ . Case 3 is trivial. Case 4 follows from (2) and  $L(x) - L(y) \leq L_i(x) - L_i(y)$ . This completes the proof. ■

As a consequence of Theorem 6, we shall show that every functional  $l \in \text{Lip}_k(V)$ , where  $V$  is a linear subspace of  $X$ , has a norm-preserving extension to all of  $X$ ; this fact is formulated in

**THEOREM 7.** *Let  $k > 0$  be fixed and  $V \subset X$  be a linear subspace, then every  $l \in \text{Lip}_k(V)$  has a norm-preserving extension  $L \in \text{Lip}_k(X)$ .*

*Proof.* We use the same construction as in the proof of Theorem 6. The mapping

$$k \cdot q: X \rightarrow \mathbb{R}$$

which is defined by

$$(k \cdot q)(x) := k \cdot q(x)$$

for every  $x \in X$  has all the properties required for  $p$  in the preceding theorem, so that it suffices to show that the extension  $L$  preserves the norm if in the construction we replace  $p$  by  $k \cdot q$ . As  $L$  is an extension, it will be enough to show that

$$\|L\|_X \leq \|l\|_V.$$

To this aim choose an arbitrary  $x \in X \setminus \{0\}$ . If  $L(x) = 0$ , then of course  $|L(x)|/q(x) \leq \|l\|_V$  and we only have to consider:

Case 1.  $L(x) > 0$ .

Case 2.  $L(x) < 0$ .

For Case 1, we have

$$\frac{|L(x)|}{q(x)} = \frac{L_s(x)}{q(x)} = \sup_{v \in V \setminus \{0\}} \frac{l(v) - k \cdot q(x - v)}{q(x)}.$$

Now, if for  $v \in V$ , we have  $q(x) < q(v)$ , then  $l(v)/q(v) \leq k$  yields

$$l(v) \cdot \frac{q(v) - q(x)}{q(v)} \leq k \cdot (q(v) - q(x)) \leq k \cdot q(x - v)$$

or

$$\frac{l(v) - k \cdot q(x - v)}{q(x)} \leq \frac{l(v)}{q(v)} \leq \|l\|_V. \tag{3}$$

If on the other hand,  $q(x) \leq q(v)$ , the last inequality is obvious.

Case 2. In this case, by the symmetry of  $L$ , we have

$$\frac{|L(x)|}{q(x)} = \frac{-L(x)}{q(x)} = \frac{L(-x)}{q(x)},$$

and if we replace  $x$  by  $-x$  get Case 1.

So inequality (3) holds for all  $x \in X$  and we have

$$\|L\|_X = \sup_{x \neq 0} \frac{|L(x)|}{q(x)} \leq \|l\|_V.$$

This completes the proof. ■

We shall not investigate the numerous consequences of the Hahn–Banach theorem in this context, but only those used in the sequel to characterize best approximations in metric linear spaces. Stating the preliminaries for the approximation theoretic considerations, we need an additional property of the quasinorm  $q$ , which ensures that the unit balls (in the metric space  $X$ ) are connected.

We call a quasinorm  $q: X \rightarrow \mathbb{R}$  monotone, if

$$q(tx) \leq q(sx)$$

holds for all  $t, s \in \mathbb{R}$  with  $|t| \leq |s|$ .

We shall now briefly sketch the proof of the following:

*Remark.* If  $X \neq \{0\}$  and the quasinorm  $q$  is monotone, then  $(\text{Lip}(X), \|\cdot\|)$  is not complete.

*Sketch of Proof.* First we note, that

$$\text{Lip}(X) = \bigcup_{k=1}^{\infty} \text{Lip}_k(X)$$

and that by the Baire category theorem it suffices to show that, for all  $k \in \mathbb{N}$ ,  $\text{Lip}_k(X)$  is closed (in  $\text{Lip}(X)$ ) and has no interior points.

Let  $k \in \mathbb{N}$  be fixed. Lemma 2 shows that  $\text{Lip}_k(X)$  is closed in  $\text{Lip}(X)$ . So let  $f \in \text{Lip}_k(X)$  and  $\varepsilon > 0$  and  $x_0 \in X \setminus \{0\}$  be chosen arbitrarily. Then define  $V := \text{span}\{x_0\}$  and choose  $1 > \delta > 0$  such that

$$q(s \cdot x_0) < \frac{\varepsilon}{4 \cdot k} \cdot q(t \cdot x_0)$$

for all  $s \in [0, \delta]$  and  $t \in [1 - \delta, 1 + \delta]$ . Then choose  $K \in \mathbb{N}$  such that

$$4 \cdot k > K > 2 \cdot k$$

and define a function  $h: V \rightarrow \mathbb{R}$  by

$$h(\alpha \cdot x_0) := \begin{cases} 0, & |\alpha| \leq 1 - \delta, \\ K \operatorname{sgn}(\alpha) q((|\alpha| - (1 - \delta)) \cdot x_0), & 1 - \delta < |\alpha| \leq 1, \\ K \operatorname{sgn}(\alpha) q(((1 + \delta) - |\alpha|) \cdot x_0), & 1 < |\alpha| < 1 + \delta, \\ 0, & |\alpha| \geq 1 + \delta. \end{cases}$$

One may then show that

- (a)  $h \notin \operatorname{Lip}_k(V)$ ,
- (b)  $\|h\|_V \leq \varepsilon$ ,
- (c)  $h \in \operatorname{Lip}_{2k}(V)$ .

By Theorem 7,  $h$  has a norm-preserving extension  $H$  to all of  $X$  and hence the function  $f_1 := f + H$  is in  $\operatorname{Lip}(X)$  and we have

$$\|f - f_1\| = \|H\| \leq \varepsilon.$$

But on the other hand

$$|f_1(x_0) - f_1((1 - \delta) \cdot x_0)| \geq k \cdot q(x_0 - (1 - \delta) \cdot x_0),$$

so that  $f_1 \notin \operatorname{Lip}_k(X)$ , which means that  $f$  cannot be an interior point. This finishes the proof of the remark. ■

In the sequel  $q$  is always assumed to be a monotone quasinorm. If  $V \subset X$  is a subset of  $X$  and  $x \in X$ , we define

$$q(x - V) := \inf_{v \in V} q(x - v);$$

then we have

**LEMMA 8.** *If  $V \subset X$  is a subspace and  $x \in X \setminus \bar{V}$ , then there is a functional  $l \in \operatorname{Lip}_2(X)$  such that*

- (a)  $\|l\| = 1$ ,
- (b)  $l(x) = q(x - V)$ ,
- (c)  $l(x) = l(x + v)$  for all  $v \in V$ .

*Proof.* Let  $V_0$  be the sum of  $V$  and the one-dimensional linear space spanned by  $x$ , i.e.,

$$V_0 := \{v + \alpha x; v \in V, \alpha \in \mathbb{R}\}.$$

We then define  $l_0: V_0 \rightarrow \mathbb{R}$  by

$$l_0(v + \alpha x) := (\operatorname{sgn}(\alpha)) \cdot q(\alpha x - V).$$

$l_0$  is continuous and (b) and (c) hold for  $l = l_0$ . Further

$$-l_0(v + \alpha x) = l_0(-(v + \alpha x))$$

for all  $v \in V$  and  $\alpha \in \mathbb{R}$  and

$$\sup \left\{ \frac{|l_0(v + \alpha x)|}{q(v + \alpha x)}; v + \alpha x \neq 0 \right\} = \sup \left\{ \frac{q(\alpha x - V)}{q(\alpha x + v)}; v + \alpha x \neq 0 \right\} \leq 1$$

so that

$$\|l_0\|_{V_0} = 1.$$

We now show that for all  $u, v \in V$  and  $\alpha, \beta \in \mathbb{R}$

$$|l_0(u + \alpha x) - l_0(v + \beta x)| \leq 2 \cdot q((u + \alpha x) - (v + \beta x))$$

holds, i.e.,  $l_0$  is in  $\operatorname{Lip}_2(V_0)$ . To this aim we consider five cases.

*Case 1.* Either  $\alpha = 0$  or  $\beta = 0$ .

Without loss of generality we may assume  $\alpha = 0$ , then we have

$$|l_0(u + \alpha x) - l_0(v + \beta x)| \leq q(\beta x - V) \leq q((\beta x + v) - (\alpha x + u)).$$

*Case 2.*  $\alpha > 0$  and  $\beta > 0$ .

Let  $\varepsilon > 0$  be given and choose  $w_\varepsilon \in V$  such that

$$q(\beta x + w_\varepsilon) \leq q(\beta x - V) + \varepsilon,$$

then

$$\begin{aligned} l_0(u + \alpha x) - l_0(v + \beta x) &= q(\alpha x - V) - q(\beta x - V) \\ &\leq q(\alpha x + (u - v + w_\varepsilon)) - q(\beta x + w_\varepsilon) + \varepsilon \\ &\leq q((\alpha x + u) - (\beta x + v)) + \varepsilon; \end{aligned}$$

by the same argument one shows that

$$l_0(v + \beta x) - l_0(u + \alpha x) \leq q((\alpha x + u) - (\beta x + v)) + \varepsilon$$

holds and since  $\varepsilon > 0$  was arbitrary the result follows.

*Case 3.*  $\alpha < 0$  and  $\beta < 0$ .

This case can be treated as in Case 2.

Case 4.  $\alpha > 0$  and  $\beta < 0$ .

Since  $q$  is monotone we have

$$\begin{aligned} |l_0(v + \alpha x) - l_0(u + \beta x)| &= q(\alpha x - V) + q(|\beta| x - V) \\ &\leq q((\alpha + |\beta|) x - V) + q((|\beta| + \alpha) x - V) \\ &\leq 2 \cdot q((\alpha x + u) - (\beta x + v)). \end{aligned}$$

Case 5.  $\alpha < 0$  and  $\beta > 0$ .

This can be treated as in Case 4.

We have constructed  $l_0 \in \text{Lip}_2(V_0)$  which satisfies (a), (b), and (c) and by Theorem 7,  $l_0$  has a norm preserving extension to all of  $X$ . This completes the proof. ■

Setting  $V := \{0\}$  in the preceding lemma, we have a corollary which shows that the nonlinear dual space contains “enough” functionals.

COROLLARY 9. *If  $k \geq 2$ , then*

$$q(x) = \max\{l(x); l \in B_k(0, 1)\} \quad \text{for all } x \in X.$$

As in the theory of linear dual spaces, we can describe the quasinorm via the extremal functionals of  $B_k(0, 1)$ . This will be a consequence of

THEOREM 10. *If  $V$  is a linear subspace of  $X$ , then every functional  $l$  which is an extreme point of the unit ball in  $\text{Lip}_k(V)$  has an extremal extension  $L \in \text{ext}(B_k(0, 1))$ .*

*Proof.* Let  $E(l)$  be the set of all norm-preserving extensions of  $l$  to all of  $X$ .  $E(l)$  is nonempty by Theorem 7 and convex and we shall show that  $E(l)$  is an extremal subset of  $B_k(0, 1)$ .

For if  $L_1, L_2 \in B_k(0, 1)$ ,  $\lambda \in (0, 1)$  and

$$L := \lambda L_1 + (1 - \lambda) L_2 \in E(l),$$

it follows from  $L|_V = l$  and  $l$  being extremal that

$$L_{1|V} = L_{2|V} = l;$$

since  $\|L_1\|_V = \|L_2\|_V = \|L_1\|_X = \|L_2\|_X = \|l\|_V$  we have

$$L_1, L_2 \in E(l).$$

Since pointwise limits of nets  $(l_\delta, \delta \in D) \subset E(l)$  are again in  $E(l)$ , this shows that  $E(l)$  is a closed subset of  $B_k(0, 1)$  relative to the topology  $T_k$

and hence by Corollary 5 compact, so that by Theorem 3 it has extreme points. This completes the proof. ■

It is easy to see that if  $x \in X \setminus \{0\}$  and  $V := \text{span}\{x\}$ , the space spanned by  $x$ , then for  $k \geq 2$  the functional  $l: V \rightarrow \mathbb{R}$  defined by

$$l(\alpha x) := \text{sgn}(\alpha) \cdot q(\alpha x)$$

is an extreme functional of the unit ball in  $\text{Lip}_k(V)$ . As a consequence of the preceding theorem,  $l$  has an extension to an element of  $\text{ext}(B_k(0, 1))$ . We therefore obtain

COROLLARY 11. *If  $k \geq 2$ , then*

$$q(x) = \max\{l(x); l \in \text{ext}(B_k(0, 1))\} \quad \text{for all } x \in X.$$

### III. CHARACTERIZATION OF BEST APPROXIMATIONS

The approximation problem in a metric linear space considered here is posed as follows:

If  $V \subset X$  is a linear subspace and  $x \in X \setminus \bar{V}$ , we are looking for elements  $\bar{v} \in V$  such that

$$q(x - \bar{v}) = q(x - V). \quad (4)$$

Every element  $\bar{v} \in V$  that satisfies (4) is called a best approximation (to  $x$  from  $V$  in the quasinorm  $q$ ); the set of all these best approximations is denoted by

$$P_V(x, q)$$

which may be empty of course even when  $V$  is finite dimensional.

In this chapter, we give characterizations of best approximations (in the above sense) by elements of the nonlinear dual space  $\text{Lip}_k(X)$ . The characterizing properties and their proofs are very similar to the ones known from the theory of normed linear spaces (cf. [3, 6, 12]) and they show that the nonlinear dual space introduced here in many ways behaves like the linear dual space of a normed linear space.

It remains to be seen, however, what the functionals and the extreme functionals in particular look like when we consider special metric linear spaces. As far as it is known to the author this has only been done in very simple cases (cf. [10]).

The first characterization presented here is a simple corollary to Lemma 8.

**THEOREM 12.** *If  $V \subset X$  is a linear subspace and  $k \geq 2$ , then  $\bar{v} \in V$  is a best approximation to  $x \in X \setminus \bar{V}$  if and only if there exists a functional  $l \in \text{Lip}_k(X)$  such that*

- (a)  $\|l\| = 1$ ,
- (b)  $l(x - \bar{v}) = q(x - \bar{v})$ ,
- (c)  $l(x) = l(x + v)$  for all  $v \in V$ .

*Proof.* If this functional exists, then

$$q(x - \bar{v}) = l(x - \bar{v}) = l(x - v) \leq \|l\| q(x - v) = q(x - v)$$

shows that  $\bar{v}$  is a best approximation. The necessity part of the proof is a trivial consequence of Lemma 8 and hence omitted. ■

In normed linear spaces a best approximation can be characterized by (a), (b), and (c) of Theorem 12, where  $l$  is a continuous linear functional (cf. [12, p. 18]). The next proposition will show that in many cases of quasinormed spaces there is no linear functional to characterize best approximations in this way. We therefore define the set

$$T_V^k(x) := \{l: X \rightarrow \mathbb{R}; \|l\| = 1, l(x) = q(x - V), l(x) = l(x + v) \text{ for all } v \in V\}$$

which is easily seen to be convex. With this definition we have

**PROPOSITION 13.** *Let  $k \geq 2$  and  $V \subset X$  be a linear subspace. Assume that for every  $y \in X \setminus \{0\}$  there is a real number  $t = t(y) > 0$  such that*

$$t \cdot q(y) > q(t \cdot y).$$

*Then for every  $x \in X \setminus \bar{V}$  with  $P_V(x, q) \neq \emptyset$ ,  $T_V^k(x)$  does not contain a linear functional.*

*Proof.* Let  $x \in X \setminus \bar{V}$  with  $P_V(x, q) \neq \emptyset$  and  $\bar{v} \in P_V(x, q)$ . Then suppose  $l$  is a linear functional in  $T_V^k(x)$ . Since  $l(v) = 0$  for all  $v \in V$  and  $l(x) = q(x - \bar{v}) > 0$ , we have for  $t = t(x - \bar{v}) > 0$  that

$$q(t(x - \bar{v})) \geq l(t(x - \bar{v})) = t \cdot l(x) = t \cdot q(x - \bar{v})$$

which is a contradiction to our assumption. ■

Note that the assumption in Proposition 13 is satisfied for every bounded quasinorm and in particular for every  $p$ -norm ( $0 < p < 1$ ) (cf. [9, Satz 3.3]).

Another important characterization of best approximations in the theory of normed linear spaces uses extreme functionals of the dual unit ball and is due to Garkavi [6]; these are also convenient in the usual concrete

spaces because for these spaces the general form of the extremal points is well known and simple. The characterizations can be generalized to metric linear spaces, but in this case the extreme functionals are only known for very simple examples. We define for  $k \geq 2$  and  $y \in X$ ,

$$E_k(y) := \{l \in \text{ext}(B_k(0, 1)); l(y) = q(y)\}$$

the set of peaking extreme functionals, which by Corollary 11 is always nonempty. By making natural modifications in the proof of Garkavi's theorem as given in [4, Theorem 3.9], we then have

**THEOREM 14.** *If  $V \subset X$  is a linear subspace and  $k \geq 2$  and  $x \in X \setminus \bar{V}$ , then the following three statements are equivalent:*

(A)  $\bar{v} \in P_V(x, q)$ .

(B) *To every  $v \in V$  there is a functional  $l = l_v \in E_k(x - \bar{v})$  such that  $l(x - v) - l(x - \bar{v}) \geq 0$ .*

(C)  $\max\{l(x - v) - l(x - \bar{v}); l \in E_k(x - \bar{v})\} \geq 0$  for all  $v \in V$ .

*Proof.* (A)  $\rightarrow$  (B) Let  $\bar{v} \in P_V(x, q)$  and  $v \in V$  be chosen arbitrarily. Then we define the set of peaking functionals

$$P := \{l \in B_k(0, 1); l(x - \bar{v}) = q(x - \bar{v})\}$$

and the set

$$P_v := \{l \in P; l(x - v) - l(x - \bar{v}) = \sup_{l_0 \in P} \{l_0(x - v) - l_0(x - \bar{v})\}\}.$$

By Corollary 9,  $P$  is always nonempty, further it is convex and closed relative to the topology  $T_k$  and hence by Corollary 5 compact.

Also  $l_1, l_2 \in B_k(0, 1)$ ,  $\lambda \in (0, 1)$  and  $l := \lambda l_1 + (1 - \lambda) l_2 \in P$  together with

$$q(x - \bar{v}) = l(x - \bar{v}) = \lambda l_1(x - \bar{v}) + (1 - \lambda) l_2(x - \bar{v}) \leq q(x - \bar{v})$$

imply

$$l_1, l_2 \in P$$

and thus  $P$  is an extremal subset of  $B_k(0, 1)$ .

If we define

$$F: B_k(0, 1) \rightarrow \mathbb{R}$$

by

$$F(l) := l(x - v) - l(x - \bar{v}),$$



then  $F$  is continuous relative to the topology  $T_k$  and if  $l_1, l_2 \in B_k(0, 1)$  and  $\lambda \in [0, 1]$ , then

$$F(\lambda l_1 + (1 - \lambda) l_2) = \lambda F(l_1) + (1 - \lambda) F(l_2).$$

Further

$$P_v = \{l \in P; F(l) = \max_{l_0 \in P} F(l_0)\}$$

is convex, compact (relative  $T_k$ ) and an extremal subset of  $P$ , since  $L_1, L_2 \in P$ ,  $\lambda \in (0, 1)$  and  $L := \lambda L_1 + (1 - \lambda) L_2 \in P_v$  together with

$$F(L) = \lambda F(L_1) + (1 - \lambda) F(L_2) \leq F(L)$$

imply

$$L_1, L_2 \in P_v.$$

By Theorem 3  $\text{ext}(P)$  and  $\text{ext}(P_v)$  are nonempty and  $F$  attains its maximum on  $P$  at an extreme point of  $P$ . Hence there exists a functional

$$l \in \text{ext}(P) \subset \text{ext}(B_k(0, 1))$$

such that

$$l(x - v) - l(x - \bar{v}) \geq l_0(x - v) - l_0(x - \bar{v}) \quad \text{for all } l_0 \in P.$$

Together with the definition of  $P$  it follows that  $l \in E_k(x - \bar{v})$ , but by Theorem 12, there is a functional  $l_0 \in P$  with

$$l_0(x - v) - l_0(x - \bar{v}) = 0$$

from which  $l(x - v) - l(x - \bar{v}) \geq 0$  is evident.

(B)  $\rightarrow$  (C) From (B), it follows immediately that

$$\sup\{l(x - v) - l(x - \bar{v}); l \in E_k(x - \bar{v})\} \geq 0$$

and it remains to be shown that this supremum is actually attained. But if we define  $F$ ,  $P$ , and  $P_v$  as in the proof of (A)  $\rightarrow$  (B), this follows immediately from the fact that  $P_v$  is convex and compact.

(C)  $\rightarrow$  (A) Let  $v \in V$  be chosen arbitrary and  $l \in E_k(x - \bar{v})$  such that

$$l(x - v) - l(x - \bar{v}) \geq 0,$$

then we have

$$q(x - \bar{v}) = l(x - \bar{v}) = l(x - v) + (l(x - \bar{v}) - l(x - v)) \leq l(x - v) \leq q(x - v)$$

and hence  $\bar{v}$  is a best approximation to  $x$  from  $V$ . This completes the proof. ■

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